# 1 Introduction

Whist is a card game that originated in Turkey, but became prominent in England. It is an international card game that has transformed into other popular card games such as Bid Whist, Spades, and Bridge [1] .

**Definition 1.1** *[2] A whist tournament, $Wh(4n+1)$, for $4n+1$ players is a schedule of games each involving two players playing against two others, such that*
*(i) the games are arranged in $4n+1$ rounds, each $n$ games,*
*(ii) each player plays in one game in all but one of the rounds,*
*(iii) each player partners every other player exactly once,*
*(iv) each player opposes every other player exactly twice.*
*A whist tournament, $Wh(4n)$, for $4n$ players is similarly defined except that the games are arranged in $4n-1$ rounds and every player plays in exactly one game every round.*

The four players in any game can be thought of as sitting around a circular table where partners sit across from each other. Partners of the first kind are defined to be partners sitting in the North South positions while partners of the second kind sit in the East West positions.

In the 1970s, it was established that whist tournaments for $4n$ and $4n+1$ players exist for all positive integers $n$. Beginning in the 1990s, mathematicians turned their focus to different specializations of whist tournaments. There are many specializations, but one of particular concern in this study is an ordered whist tournament which was first introduced in an unpublished paper by Y.Lu [3] and is defined below.

**Definition 1.2** *[4] An ordered whist tournament, $OWh(v)$, for $v$ players is a $Wh(v)$ such that*
*(i) each player opposes every other player exactly once as a partner of the first kind*
*(ii) each player opposes every other player exactly once as a partner of the second kind*

Through the work of Stephanie Costa, Norman Finizio, and Philip A. Leonard it is known that ordered whist tournaments exist for all $v = 4n+1$ and do not exist for multiples of 4 [5]. Another type of specialization is a generalized whist tournament.

**Definition 1.3** *[6] A generalized whist tournament design is a schedule of games for a tournament involving $v$ players to be played in $v$ rounds. A game involves $k$ players with teams of $t$ players competing. A round consists of $(v-1)/k$ simultaneous games, with a player playing in all but one round.*
*Every player partners every other player $t-1$ times.*
*Every player opposes every other player $k-t$ times.*

**Definition 1.4** *A generalized whist is said to be $\mathbb{Z}-$cyclic if all the players are elements of $\mathbb{Z}_v - \{0\}$, and all the games of round $i$ can be obtained by adding $i \bmod (v)$ to each player in round 0.*

When $v \equiv 1 \bmod 6$, the initial round of a $\mathbb{Z}$-cyclic generalized whist tournament is traditionally thought of as the round that omits player 0. In this paper we focused on specific generalized whist games of size 6 with teams of size 3. Using symmetric differences it follows that a collection of $n$ games $(a_i, b_i, c_i, d_i, e_i, f_i)$, $i = 1, ..., n$ form the initial round game of a $\mathbb{Z}$-cyclic generalized whist tournament on $v = 6n + 1$ players if

$$\bigcup_{i=1}^{n} \{a_i, b_i, c_i, d_i, e_i, f_i\} = \mathbb{Z}_{6n+1} - \{0\} \tag{1.1}$$

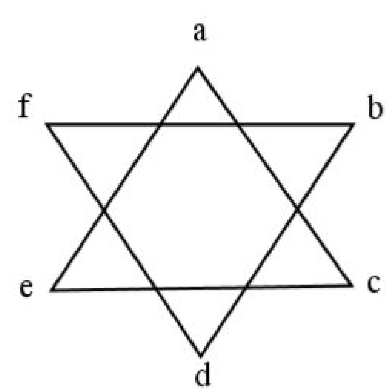
$$\bigcup_{i=1}^{n} \{\pm(a_i - c_i), \pm(a_i - e_i), \pm(c_i - e_i), \pm(b_i - d_i), \pm(b_i - f_i), \pm(d_i - f_i)\} \tag{1.2}$$

gives us two copies of $\mathbb{Z}_p - \{0\}$

$$\begin{aligned}\bigcup_{i=1}^{n} \{&a_i - b_i, a_i - d_i, a_i - f_i, d_i - a_i, d_i - c_i, d_i - e_i, b_i - c_i, b_i - a_i, b_i - e_i, \\ &e_i - b_i, e_i - d_i, e_i - f_i, c_i - b_i, c_i - d_i, c_i - f_i, f_i - a_i, f_i - c_i, f_i - e_i\}\end{aligned} \tag{1.3}$$

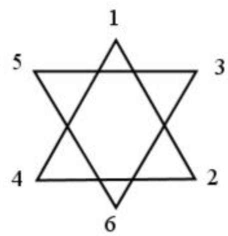
gives us three copies of $\mathbb{Z}_p - \{0\}$

We refer to the differences 1.2 and 1.3 as the partner and opponent differences, respectively. Games of this tournament are denoted by the 6-tuple $(a, b, c, d, e, f)$ where $(a, c, e)$ are partners and $(b, d, f)$ are partners. Players would sit at the table as follows:

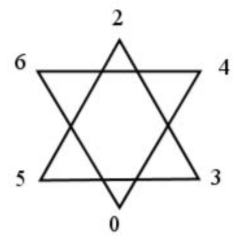


Below is an example of a $\mathbb{Z}$-cyclic generalized whist tournament with 7 players. Note, we label the round by the player who sits out.

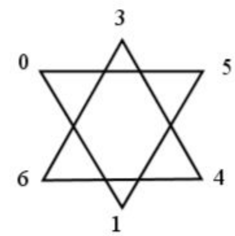
**Example 1.1**



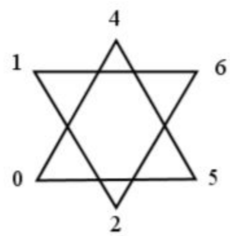

*Round 0:* $(1, 3, 2, 6, 4, 5)$



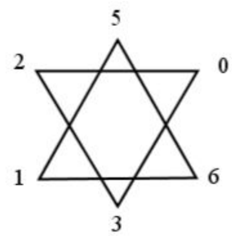

*Round 1:* $(2, 4, 3, 0, 5, 6)$



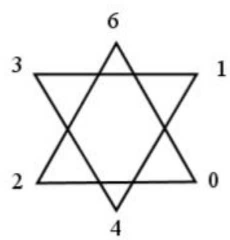

*Round 2:* $(3, 5, 4, 1, 6, 0)$



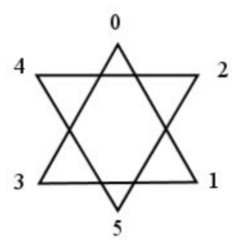

*Round 3:* $(4, 6, 5, 2, 0, 1)$




*Round 4:* $(5, 0, 6, 3, 1, 2)$




*Round 5:* $(6, 1, 0, 4, 2, 3)$



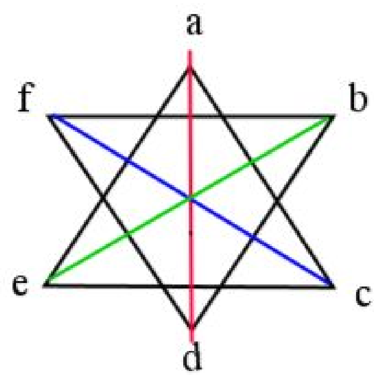

*Round 6:* $(0, 2, 1, 5, 3, 4)$

In this example, since there are seven players, there is one game in each round and there are a total of seven rounds, starting with round 0. If we look at round 0, we see that $(1, 2, 4)$ are teammates and they oppose players $(3, 6, 5)$. Since every player sits out once, we say the tournament is balanced.

# 2 Main Results

In this project, we worked to see if it would be possible to extend the idea of an ordered whist tournament to a generalized whist tournament on $6n$ or $6n + 1$

players. We focused on tournaments where the players are divided into $n$ games of size 6 each consisting of two teams of size 3. We aimed to balance the 3 occasions where the players meet as opponents. In order to introduce some notation, consider the game $(a, b, c, d, e, f)$ again where $(a, c, e)$ are partners and $(b, d, f)$ are partners, we would sit them around the table as follows:




We say a player is on Axis N if player is sitting in position $a$ or $d$
A player is on Axis E if player is sitting in position $b$ or $e$
A player is on Axis W if player is sitting in position $f$ or $c$

We define our new specialization:

**Definition 2.1** *A $(3,6)GWhD(6n+1)$ is ordered if each player opposes every other player exactly once while sitting on Axis N, Axis W, and Axis E. We denote such a tournament by $(3,6)OGWhD(6n+1)$*

To test whether the three occasions the opponents meet are balanced, we developed this new theorem. The theorem below allows us to check opponent differences to see whether the construction is ordered.

**Theorem 1** *Let G be an abelian group such that*

$$|G| = 6n + 1 \tag{2.4}$$

*Let ($a_i$,$b_i$,$c_i$,$d_i$,$e_i$,$f_i$), $0 \leq i \leq n-1$ denote non-identity elements in G. Suppose that the collection of games ($a_i$,$b_i$,$c_i$,$d_i$,$e_i$,$f_i$), $0 \leq i \leq n-1$ constitutes an initial round of a cyclic (3,6) GwhD(v). This (3,6) GWhD(v) is ordered if and only if*

$$\bigcup_{i=0}^{n-1} = \{(a_i - b_i), (a_i - d_i), (a_i - f_i), (d_i - a_i), (d_i - c_i), (d_i - e_i)\} = G - \{e\} \tag{2.5}$$

*and*

$$\bigcup_{i=0}^{n-1} = \{(b_i - a_i), (b_i - c_i), (b_i - e_i), (e_i - b_i), (e_i - d_i), (e_i - f_i)\} = G - \{e\} \quad (2.6)$$

*and*

$$\bigcup_{i=0}^{n-1} = \{(c_i - b_i), (c_i - d_i), (c_i - f_i), (f_i - a_i), (f_i - c_i), (f_i - e_i)\} = G - \{e\} \quad (2.7)$$

*where $e$ is the identity for $G$.*

*Proof:* ($\Leftarrow$) Suppose that 2.5,2.6, and 2.7 from above are true. Since G has order $6n+1$, G has $6n$ distinct non-identity elements. This means that the $6n$ differences are all unique.

Assume that the tournament is not ordered. Then there exists at least one pair $(x, y)$ having the property that in their 3 meetings as opponents, $x$, say, sits on Axis N both times. Without loss of generality, we can assume that $x$ and $y$ meet as opponents in the following 2 games:

$$(x, y, \star, \diamond, \circ, \triangleleft)(x, \dagger, \wr, y, ?, !)$$

Since these 2 games are translates of games in the initial round, it follows that

$$x - y = a_i - b_i \text{ for some initial round } (a_i, b_i, c_i, d_i, e_i, f_i)$$
$$x - y = a_j - d_j \text{ for some initial round } (a_j, b_j, c_j, d_j, e_j, f_j)$$

Therefore,
$x - y = x - y$ which means $a_i - b_i = a_j - d_j$
which contradicts the facts that differences are distinct. The above proof is similar if $x$ is sitting on Axis E or on Axis W. Thus the tournament is ordered.

($\Rightarrow$) Suppose that the $Wh(6n+1)$ is ordered.

Assume that,

$$\bigcup_{i=0}^{n-1} = \{(a_i - b_i), (a_i - d_i), (a_i - f_i), (d_i - a_i), (d_i - c_i), (d_i - e_i)\} \neq G - \{e\} \quad (2.8)$$

or

$$\bigcup_{i=0}^{n-1} = \{(b_i - a_i), (b_i - c_i), (b_i - e_i), (e_i - b_i), (e_i - d_i), (e_i - f_i)\} \neq G - \{e\} \quad (2.9)$$

or

$$\bigcup_{i=0}^{n-1} = \{(c_i - b_i), (c_i - d_i), (c_i - f_i), (f_i - a_i), (f_i - c_i), (f_i - e_i)\} \neq G - \{e\} \quad (2.10)$$

Since none of the differences can equal the identity, this assumption implies that at least two differences have the same value. However, no two differences can be equal without violating the assumption that $Wh(6n+1)$ is ordered.Therefore, if two differences are equal, they have to come from distinct initial round tables:

$$\text{Table } i = (a_i, b_i, c_i, d_i, e_i, f_i) \text{ and Table } j = (a_j, b_j, c_j, d_j, e_j, f_j)$$

Suppose that $a_i - b_i = a_j - d_j$

Define $x$ by the requirement that $a_j + x = a_i$ Then in round $x$, Table $j$ becomes

$$(a_j + x, b_j + x, c_j + x, d_j + x, e_j + x, f_j + x)$$

which is equivalent to

$$(a_i, b_j + x, c_j + x, b_i, e_j + x, f_j + x) \quad (2.11)$$

Comparing Table $i$ with 2.11 , we see that $a_i$ opposes $b_i$ as a partner sitting on Axis N at both tables which contradicts the fact that the tournament is ordered. The above proof is similar for Axis E and Axis W seating positions. Similar contradictions occur for all the other possible matchings of the differences.

We know $v$ had to be of the form $6n + 1$ and could not be of the form $6n$ because the theorem below states that for the tournament to be ordered, $v \equiv 1 \bmod 6$

**Theorem 2.1** *If a (3,6)GWhD(v) is ordered then* $v \equiv 1 \bmod 6$

*Proof:* Suppose the$(3,6)GWhD(v)$ is based on the set $X$ with $|X| = v$. Let $x \in X$ and consider the totality of games in which $x$ sits on axis $N$. Suppose there are $k$ such games. In each game $x$ opposes 3 distinct players each sitting on three different axes. Since the Wh($v$) is ordered the $3k$ players that $x$ opposes in these $k$ games must contain the totality of players in the tournament distinct from $x$. We conclude that $v = 3k + 1$. This means $v$ will always be a multiple of six plus one and therefore $v = 6n + 1$.

My construction came from the definition of a cyclotomic class:

**Definition 2.2** *Let $p$ denote a prime of the form $6n+1$. If $r$ is a generator for $\mathbb{Z}_p - \{0\}$ then for each non-zero element $x \in \mathbb{Z}_p - \{0\}$ there exists a unique integer $i$ such that when $r$ is raised to the $i^{th}$ power ($r^i$), all elements of $\mathbb{Z}_p - \{0\}$ are generated. If $6|(p-1)$, $x = r^i$, and $i \equiv j \bmod 6$ then we say $x$ is in the $j^{th}$ cyclotomic class of index 6, where $j < 6$.*

We can visualize the field of $\mathbb{Z}_p - \{0\}$, $p = 6n+1$, as being divided into six cyclotomic classes as seen in the chart below. We raise the generator, $r$, to $p$ many powers in order to get all the $6n+1$ total players in the game. There will be $n$ rows in each cyclotomic class.

$$\left.\begin{array}{cccccc}
\underline{0} & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} \\
r^0 & r^1 & r^2 & r^3 & r^4 & r^5 \\
r^6 & r^7 & r^8 & r^9 & r^{10} & r^{11} \\
r^{12} & r^{13} & r^{14} & r^{15} & r^{16} & r^{17} \\
r^{18} & r^{19} & r^{20} & r^{21} & r^{22} & r^{23} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r^{6i} & r^{6i+1} & r^{6i+2} & r^{6i+3} & r^{6i+4} & r^{6i+5} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r^{6(n-1)} & r^{6n-5} & r^{6n-4} & r^{6n-3} & r^{6n-2} & r^{6n-1}
\end{array}\right\}\text{n}$$

We have $6n+1$ total players in the tournament. There are $6n$ players in any round (since one player sits out in every round). In each round, there are $n$ games. Our goal is to find an initial round of the tournament where all players in table above are seated in exactly one game. In order to achieve this goal, we want the six players in the first game of the initial round to come from different cyclotomic classes. Once this happens, we can obtain the other games in the round by multiplying this group of players by $r^{6k}$, $1 \leq k \leq n$. This means all the other games from the same round will have players from different cyclotomic classes. We know $r$ must be in the first class. Thus, if we choose any $y$ in the third class and $z$ in the fifth class, we are guaranteed an element in classes 2,4,0,respectively, by multiplying each of those elements by any odd power of the primitive root. This motivates the construction of our initial round $(r, r^6, y, yr^5, z, zr^5) \times r^{6k}$, $0 \leq k \leq n-1$. When we multiply any game by $r^6$ it generates all the games of that round.

# 3 The Major Constructions

Let $p$ be a prime of the form $p = 6n+1$ with $n$ odd. Let $r$ denote the primitive root of $p$.

**Construction 1** $(r^6, r, r^5y, y, r^5z, z) \times r^{6k}$, $0 \leq k \leq n-1$

**Theorem 3.1** *If* $(-1+r^5) \in C_b^6$ *and*

*a. We need to have exactly one of the differences* $r-y$, $r-z$, *and* $y-z$ *in each of the sets* $C_b^6 \cup C_{b+3}^6, C_{b+1}^6 \cup C_{b+4}^6$, *and* $C_{b+2}^6 \cup C_{b+5}^6$

*b.* $r(-1+yr^4) \in C_{b+5}^6$

*c.* $r(-1+zr^4) \in C_b^6$ *or* $r(-1+zr^4) \in C_{b+3}^6$

*d.* $r^6 - z \in C_{b+3}^6$

*e.* $yr^5 - z \in C_{b+4}^6$ *or* $yr^5 - z \in C_{b+1}^6$

*f.* $y - r^5 z \in C_{b+4}^6$

*g.* $r^6 - y \in C_{b+2}^6$ *or* $r^6 - y \in C_{b+5}^6$

*then there exists a* $\mathbb{Z}$*-cyclic* $(3,6)OGWhD(6n+1)$.

**Construction 2** $(r^6, z, r^5 y, r, r^5 z, y) \times r^{6k}$, $0 \le k \le n-1$

**Theorem 3.2** *If* $(-1+r^5) \in C_b^6$ *and*

*a. We need to have exactly one of the differences* $r-y$, $r-z$, *and* $y-z$ *in each of the sets* $C_b^6 \cup C_{b+3}^6, C_{b+1}^6 \cup C_{b+4}^6$, *and* $C_{b+2}^6 \cup C_{b+5}^6$

*b.* $r(-1+yr^4) \in C_{b+2}^6$ *and* $r^6 - y \in C_{b+2}^6$

*or*

$r(-1+yr^4) \in C_{b+5}^6$ *and* $r^6 - y \in C_{b+5}^6$

*c.* $r(-1+zr^4) \in C_b^6$ *and* $r^6 - z \in C_b^6$

*or*

$r(-1+zr^4) \in C_{b+3}^6$ *and* $r^6 - z \in C_{b+3}^6$

*d.* $yr^5 - z \in C_{b+4}^6$ *and* $y - r^5 z \in C_{b+1}^6$

*or*

$yr^5 - z \in C_{b+1}^6$ *and* $y - r^5 z \in C_{b+4}^6$

*then there exists a* $\mathbb{Z}$*-cyclic* $(3,6)OGWhD(6n+1)$.

**Construction 3** $(r^5y, r, r^6, y, r^5z, z) \times r^{6k}$, $0 \le k \le n-1$

**Theorem 3.3** *If* $(-1+r^5) \in C_b^6$ *and*

*a. We need to have exactly one of the differences* $r-y$, $r-z$, *and* $y-z$ *in each of the sets* $C_b^6 \cup C_{b+3}^6, C_{b+1}^6 \cup C_{b+4}^6$, *and* $C_{b+2}^6 \cup C_{b+5}^6$

*b.* $r(-1+yr^4) \in C_{b+4}^6$ *and* $y - r^5z \in C_{b+5}^6$

*or*

$r(-1+yr^4) \in C_{b+5}^6$ *and* $y - r^5z \in C_{b+4}^6$

*c.* $r(-1+zr^4) \in C_b^6$ *or* $r(-1+zr^4) \in C_{b+3}^6$

*d.* $r^6 - z \in C_b^6$ *or* $r^6 - z \in C_{b+3}^6$

*e.* $yr^5 - z \in C_{b+2}^6$ *and* $r^6 - y \in C_{b+4}^6$

*or*

$yr^5 - z \in C_{b+1}^6$ *and* $r^6 - y \in C_{b+5}^6$

*then there exists a* $\mathbb{Z}$*-cyclic* $(3,6)OGWhD(6n+1)$.

**Construction 4** $(r^6, y, r^5y, z, r^5z, r) \times r^{6k}$, $0 \le k \le n-1$

**Theorem 3.4** *If* $(-1+r^5) \in C_b^6$ *and*

*a. We need to have exactly one of the differences* $r-y$, $r-z$, *and* $y-z$ *in each of the sets* $C_b^6 \cup C_{b+3}^6, C_{b+1}^6 \cup C_{b+4}^6$, *and* $C_{b+2}^6 \cup C_{b+5}^6$

*b.* $r(-1+yr^4) \in C_{b+5}^6$ *or* $r(-1+yr^4) \in C_{b+2}^6$

*c.* $r(-1+zr^4) \in C_{b+3}^6$

*d.* $r^6 - z \in C_b^6$ *or* $r^6 - z \in C_{b+3}^6$

*e.* $yr^5 - z \in C_{b+1}^6$

*f.* $r^6 - y \in C_{b+5}^6$

*g.* $y - r^5z \in C_{b+1}^6$ *or* $y - r^5z \in C_{b+4}^6$

*then there exists a* $\mathbb{Z}$*-cyclic* $(3,6)OGWhD(6n+1)$.

**Construction 5** $(r^6, y, r^5y, r, r^5z, z) \times r^{6k}$, $0 \le k \le n-1$

**Theorem 3.5** *If* $(-1+r^5) \in C_b^6$ *and*

*a. We need to have exactly one of the differences* $r-y$*,* $r-z$*, and* $y-z$ *in each of the sets* $C_b^6 \cup C_{b+3}^6, C_{b+1}^6 \cup C_{b+4}^6$*, and* $C_{b+2}^6 \cup C_{b+5}^6$

*b.* $r(-1+yr^4) \in C_b^6$ *and* $r^6 - z \in C_{b+2}^6$

*or*

$r(-1+yr^4) \in C_{b+5}^6$ *and* $r^6 - z \in C_{b+3}^6$

*c.* $r(-1+zr^4) \in C_{b+3}^6$ *and* $r^6 - y \in C_{b+5}^6$

*or*

$r(-1+zr^4) \in C_{b+2}^6$ *and* $r^6 - y \in C_b^6$

*d.* $yr^5 - z \in C_{b+1}^6$ *or* $yr^5 - z \in C_{b+4}^6$

*e.* $y - r^5z \in C_{b+1}^6$ *or* $y - r^5z \in C_{b+4}^6$

*then there exists a* $\mathbb{Z}$*-cyclic* $(3,6)OGWhD(6n+1)$*.*

**Construction 6** $(r^6, z, r^5y, y, r^5z, r) \times r^{6k}$, $0 \le k \le n-1$

**Theorem 3.6** *If* $(-1+r^5) \in C_b^6$ *and*

*a. We need to have exactly one of the differences* $r-y$*,* $r-z$*, and* $y-z$ *in each of the sets* $C_b^6 \cup C_{b+3}^6, C_{b+1}^6 \cup C_{b+4}^6$*, and* $C_{b+2}^6 \cup C_{b+5}^6$

*b.* $r(-1+zr^4) \in C_{b+4}^6$ *and* $yr^5 - z \in C_b^6$

*or*

$r(-1+zr^4) \in C_{b+3}^6$ *and* $yr^5 - z \in C_{b+1}^6$

*c.* $r(-1+yr^4) \in C_{b+2}^6$ *or* $r(-1+yr^4) \in C_{b+5}^6$

*d.* $r^6 - y \in C_{b+2}^6$ *or* $r^6 - y \in C_{b+5}^6$

*e.* $r^6 - z \in C_{b+3}^6$ *and* $y - r^5z \in C_{b+4}^6$

*or*

$r^6 - z \in C^6_{b+4}$ *and* $y - r^5 z \in C^6_{b+3}$

*then there exists a* $\mathbb{Z}$*-cyclic* $(3,6)OGWhD(6n+1)$.

*Proof:* In order to show that this construction produces a $\mathbb{Z}$-cyclic $(3,6)OGWhD(v)$ we must show conditions 1.2, and 2.5-2.7 are satisfied. Since $p$ is prime we are guaranteed $\mathbb{Z}_p - \{0\}$ has a generator. We also know that for any generator $r$ $r^{\frac{p-1}{2}} \equiv -1(modp)$. For us, $p = 6n+1$ so, $r^{\frac{(6n+1)-1}{2}} = r^{3n} \equiv -1(modp)$. If $n$ is even, $-1$ is in the 0 class. When $-1$ is in the 0 class, it forces two of the opponent differences to be in the same class and thus our construction will not be ordered. We want $n$ to be odd to allow $-1$ in the $3rd$ class, which means we will not have two opponent differences in the same cyclotomic class on the same axis. This allows the construction to be ordered. Therefore, $n$ must be odd so our construction can be ordered.

All the partners are the same for constructions 1-6. Therefore, for all six we must show that the partner differences produce two copies of $\mathbb{Z}_p - \{0\}$. The partner differences are $r^5(r-y)$, $r^5(r-z)$, $r^5(y-z)$, $-r^5(r-y)$, $-r^5(r-z)$, $-r^5(y-z)$, $r-y$, $r-z$, $y-z$, $-r+y$, $-r+z$, $-y+z$. We need to have exactly one of the differences $r-y$, $r-z$, and $y-z$ in $C^6_b \cup C^6_{b+3}, C^6_{b+1} \cup C^6_{b+4}$, and $C^6_{b+2} \cup C^6_{b+5}$ which is satisfied by condition a. Now we must look at the opponent differences for constructions 1-6. To check the opponent differences, we must verify conditions 2.5-2.7 for each construction.

**Construction 1**

The Axis N opponent differences are $r^6 - y$, $-r^6 + y$, $r(-1+r^5)$,$-y(-1+r^5)$, $r^6 - z$, $y - r^5 z$. By Theorem 3.1, these differences are in classes $b+2$ or $b+5$; $b+5$ or $b+2$; $b+1$; $b$; $b+3$ and $b+4$, respectively. Thus, this construction satisfies condition 2.5.

Taking the opponent differences for Axis E, we obtain $-r(-1+r^5)$, $r(-1+r^4 y)$, $-r(-1+r^4 z)$, $r(-1+r^4 z)$, $-y+r^5 z$, $z(-1+r^5)$. By Theorem 3.1, these differences are in classes $b+4$; $b+2$; $b$ or $b+3$; $b+3$ or $b$; $b+1$; and $b+5$, respectively. Thus, this construction satisfies condition 2.6.

If we do the same for Axis W opponent differences, we obtain $r(-1+r^4 y)$, $y(-1+r^5)$, $r^5 y - z$, $-r^5 y + z$, $-r^6 + z$, $-z(-1+r^5)$. By Theorem 3.1, these differences are in classes $b+5$; $b+3$; $b+4$ or $b+1$; $b+1$ or $b+4$; $b$; and $b+2$, respectively. Thus, this construction satisfies condition 2.7.

**Construction 2**

The Axis N opponent differences are $r^6 - z$, $-r(-1+r^4 z)$, $r^6 - y$, $-r(-1+r^4 y)$, $r(-1+r^5)$, $-r(-1+r^5)$. By Theorem 3.2, these differences are in classes $b$ or $b+3$; $b+3$ or $b$; $b+2$ or $b+5$; $b+5$ or $b+2$; $b+1$; and $b+4$, respectively. Thus, this construction satisfies condition 2.5.

Doing the same for Axis E partner differences we obtain $-r^6 + z$, $r(-1+r^4 z)$, $-r^5 y + z$, $-y + r^5 z$, $-z(-1+r^5)$, $z(-1+r^5)$. By Theorem 3.2, these differences are in classes $b+3$ or $b$; $b$ or $b+3$; $b+1$ or $b+4$; $b+4$ or $b+1$; $b+2$; and $b+5$,

respectively. Thus, this construction satisfies condition 2.6.
Now looking at the Axis W opponent differences we obtain $r^5y - z$, $y - r^5z$, $r(-1 + r^4y)$, $r^6 + y$, $y(-1 + r^5)$, $-y(-1 + r^5)$. By Theorem 3.2, these differnces are in classes $b + 4$ or $b + 1$; $b + 1$ or $b + 4$; $b + 2$ or $b + 5$; $b + 5$ or $b + 2$; $b + 3$; and $b$, respectively. Thus, this construction satisfies condition 2.7.

**Construction 3**

The Axis N opponent differences are $r(-1 + r^4y)$, $y - r^5z$, $r^5y - z$, $-r^6 + y$, $y(-1 + r^5)$, $-y(-1 + r^5)$. By Theorem 3.3, these differences are in classes $b + 4$ or $b + 5$; $b + 5$ or $b + 4$; $b + 2$ or $b + 1$; $b + 1$ or $b + 2$; $b + 3$; and $b$, respectively. Thus, this construction satisfies condition 2.5.
For Axis E opponent differences we obtain $-r(-1+r^4y)$, $-y+r^5z$, $r(-1+r^4z)$, $-r(-1 + r^4z)$, $-r(-1 + r^5)$, $z(-1 + r^5)$. By Theorem 3.3, these differences are in classes $b + 1$ or $b + 2$; $b + 2$ or $b + 1$; $b$ or $b + 3$; $b + 3$ or $b$; $b + 4$; and $b + 5$, respectively. Thus, this construction satisfies condition 2.6.
Looking at Axis W opponent differences we obtain $r^6 - y$, $-r^5y + z$, $r^6 - z$, $-r^6 + z$, $r(-1 + r^5)$, $-z(-1 + r^5)$. By Theorem 3.3, these differences are in classes $b + 4$ or $b + 5$; $b + 5$ or $b + 4$; $b$ or $b + 3$; $b + 3$ or $b$; $b + 1$; and $b + 2$, respectively. Thus, this construction satisfies condition 2.7.

**Construction 4**

The Axis N opponent differences are $r^6 - y$, $r(-1 + r^5)$, $-r^5y + z$, $-z(-1 + r^5)$, $r^6 - z$, $-r^6 + z$. By Theorem 3.4, these differences are in classes $b + 5$; $b + 1$; $b + 4$; $b + 2$; $b$ or $b + 3$; and $b + 3$ or $b$, respectively. Thus, this construction satisfies condition 2.5.
If we look at the Axis E opponent differences we obtain $-r^6 + y$, $-y(-1 + r^5)$, $z(-1 + r^5)$, $r(-1 + r^4z)$, $y - r^5z$, $-y + r^5z$. By Theorem 3.4, these differences are in classes $b + 2$; $b$; $b + 5$; $b + 3$; $b + 4$ or $b + 1$; and $b + 1$ or $b + 4$, respectively. Thus, this construction satisfies condition 2.6.
Now when we look at the Axis W opponent differences we obtain $y(-1 + r^5)$, $r^5y - z$, $-r(-1 + r^5)$, $-r(-1 + r^4z)$, $r(-1 + r^4y)$, $-r(-1 + r^4y)$. By Theorem 3.4, these differences are in classes $b + 3$; $b + 1$; $b + 4$; $b$; $b + 5$ or $b + 2$; and $b + 2$ or $b + 5$, respectively. Thus, this construction satisfies condition 2.7.

**Construction 5**

The Axis N opponent differences are $r^6 - y$, $-r(-1+r^4z)$, $r^6 - z$, $-r(-1+r^4y)$, $r(-1 + r^5)$, $-r(-1 + r^5)$. By Theorem 3.5, these differences are in classes $b + 5$ or $b$; $b$ or $b + 5$; $b + 2$ or $b + 3$; $b + 3$ or $b + 2$; $b + 1$; and $b + 4$, respectively. Thus, this construction satisfies condition 2.5.
For the Axis E opponent differences we obtain $-r^6 + y$, $r(-1 + r^4z)$, $y - r^5z$, $-y + r^5z$, $-y(-1 + r^5)$, $z(-1 + r^5)$. By Theorem 3.5, these differences are in classes $b + 2$ or $b + 3$; $b + 3$ or $b + 2$; $b + 1$ or $b + 4$; $b + 4$ or $b + 1$; $b$; and $b + 5$, respectively. Thus, this construction satisfies condition 2.6.
For the Axis W opponent differences we obtain $r(-1 + r^4y)$, $-r^6 + z$, $r^5y - z$, $-r^5y + z$, $y(-1 + r^5)$, $-z(-1 + r^5)$. By Theorem 3.5, these differences are in classes $b$ or $b + 5$; $b + 5$ or $b$; $b + 1$ or $b + 4$; $b + 4$ or $b + 1$; $b + 3$; and $b + 2$,

respectively. Thus, this construction satisfies condition 2.7.

**Construction 6**

The Axis N opponent differences are $r^6-z$, $y-r^5z$, $r^6-y$, $-r^6+y$, $r(-1+r^5)$, $-y(-1+r^5)$. By Theorem 3.6, these differences are in classes $b+3$ or $b+4$; $b+4$ or $b+3$; $b+2$ or $b+5$; $b+5$ or $b+2$; $b+1$; and $b$, respectively. Thus, this construction satisfies condition 2.5.

Now we looked at Axis E opponent differences to obtain $-r^6+z$, $-y+r^5z$, $-r^5y+z$, $r(-1+r^4z)$, $-z(-1+r^5)$, $z(-1+r^5)$. By Theorem 3.6, these differences are in classes $b$ or $b+1$; $b+1$ or $b$; $b+3$ or $b+4$; $b+4$ or $b+3$; $b+2$; and $b+5$, respectively. Thus, this construction satisfies condition 2.6.

Then we looked at the Axis W opponent differences to obtain $r^5y-z$, $-r(-1+r^4z)$, $r(-1+r^4y)$, $-r(-1+r^4y)$, $y(-1+r^5)$, $-r(-1+r^5)$. By Theorem 3.6, these differences are in classes $b$ or $b+1$; $b+1$ or $b$; $b+2$ or $b+5$; $b+5$ or $b+2$; $b+3$; and $b+4$, respectively. Thus, this construction satisfies condition 2.7.

The following is an example of the smallest and first known $(3,6)OGWhD(6n+1)$.

**Example 3.1** *The initial round of a* $(3,6)OGWhD(31)$*:*

$$(16,3,18,15,28,13)\times 3^{6k} \quad 0\le k\le 4$$

The initial round has all players, but player 0 playing. We can visualize the players of the initial round in each of the six cyclotomic classes as follows:

| <u>0</u> | <u>1</u> | <u>2</u> | <u>3</u> | <u>4</u> | <u>5</u> |
|---|---|---|---|---|---|
| 1 | **3** | 9 | 27 | 19 | 26 |
| **16** | 17 | 20 | 29 | 25 | **13** |
| 8 | 24 | 10 | 30 | **28** | 22 |
| 4 | 12 | 5 | **15** | 14 | 11 |
| 2 | 6 | **18** | 23 | 7 | 21 |

This initial round is generated using construction 1. In this particular case $v=31$, the primitive root, $r$, is 3, $y$ is 15, and $z$ is 13; it can be verified that these values satisfy the conditions of Theorem 3.1. The numbers in bold are the players in the first game of the initial round. In order to get to the next game of the initial round, we multiply these players by $3^6$. This means the second game in the initial round are the players below each of the red colored players. This pattern continues and allows all the players in a specific game to come from a different cyclotomic class. It also ensures that all the players, except player 0, are playing in round 0. We found many more examples of $(3,6)OGWhD(6n+1)$. A partial list for the first five hundred odd values of $n$ is below: The solutions are of the form (p, r, y, z, c) where c represents the construction number.

$(31, 3, 15, 13, 1)$
$(79, 34, 71, 53, 2)$
$(103, 77, 22, 12, 6)$
$(139, 119, 133, 26, 3)$
$(151, 133, 28, 141, 2)$
$(163, 42, 86, 108, 4)$
$(163, 94, 127, 107, 3)$
$(163, 52, 98, 80, 1)$
$(199, 127, 101, 183, 5)$
$(199, 146, 59, 170, 6)$
$(211, 155, 104, 57, 3)$
$(211, 155, 18, 160, 4)$
$(223, 180, 13, 35, 1)$
$(223, 180, 13, 35, 3)$
$(223, 180, 13, 35, 5)$
$(271, 142, 192, 94, 5)$
$(271, 269, 191, 43, 6)$
$(283, 206, 279, 123, 5)$
$(283, 154, 267, 145, 1)$
$(283, 258, 122, 26, 6)$
$(307, 98, 91, 200, 6)$
$(307, 263, 38, 279, 2)$
$(307, 151, 202, 195, 3)$
$(331, 278, 12, 41, 2)$
$(331, 90, 73, 315, 6)$
$(367, 282, 235, 305, 6)$
$(367, 282, 366, 265, 5)$
$(367, 116, 233, 294, 2)$
$(379, 153, 349, 172, 6)$
$(379, 201, 241, 299, 5)$
$(379, 46, 340, 272, 1)$
$(439, 15, 309, 241, 5)$
$(439, 15, 378, 236, 6)$
$(439, 395, 314, 34, 2)$
$(463, 3, 7, 281, 1)$
$(463, 214, 129, 332, 4)$
$(463, 214, 71, 19, 5)$
$(487, 3, 236, 457, 1)$
$(487, 3, 438, 366, 4)$
$(487, 239, 309, 415, 3)$
$(499, 340, 425, 86, 5)$
$(499, 340, 381, 102, 6)$
$(499, 321, 13, 153, 1)$
$(523, 128, 445, 132, 6)$
$(523, 128, 65, 380, 5)$
$(523, 479, 3, 392, 3)$
$(547, 407, 101, 39, 6)$
$(547, 407, 352, 418, 5)$
$(547, 339, 323, 432, 1)$
$(571, 3, 27, 243, 3)$
$(571, 91, 26, 246, 6)$
$(571, 91, 343, 298, 4)$
$(607, 3, 156, 306, 6)$
$(607, 3, 216, 502, 5)$
$(607, 510, 156, 453, 3)$
$(619, 2, 336, 108, 4)$

$(79, 3, 69, 7, 3)$
$(103, 12, 37, 54, 1)$
$(139, 130, 59, 98, 6)$
$(139, 26, 133, 119, 2)$
$(151, 54, 67, 30, 3)$
$(163, 42, 125, 124, 5)$
$(163, 94, 110, 107, 6)$
$(199, 127, 171, 73, 4)$
$(199, 179, 17, 156, 1)$
$(211, 149, 206, 131, 6)$
$(211, 155, 104, 57, 5)$
$(223, 20, 209, 67, 6)$
$(223, 180, 13, 35, 2)$
$(223, 180, 13, 35, 4)$
$(271, 142, 192, 94, 3)$
$(271, 142, 30, 201, 4)$
$(283, 206, 33, 145, 4)$
$(283, 272, 212, 82, 3)$
$(283, 154, 267, 145, 2)$
$(307, 5, 303, 139, 1)$
$(307, 263, 3, 214, 5)$
$(307, 263, 136, 236, 4)$
$(331, 3, 235, 93, 3)$
$(331, 90, 275, 315, 5)$
$(331, 90, 73, 210, 4)$
$(367, 282, 138, 115, 4)$
$(367, 42, 141, 330, 1)$
$(367, 139, 318, 194, 3)$
$(379, 317, 356, 74, 3)$
$(379, 201, 133, 78, 4)$
$(379, 279, 229, 154, 2)$
$(439, 15, 293, 184, 4)$
$(439, 238, 437, 410, 3)$
$(439, 197, 358, 323, 1)$
$(463, 214, 129, 332, 3)$
$(463, 214, 339, 93, 6)$
$(463, 295, 455, 176, 2)$
$(487, 3, 236, 457, 6)$
$(487, 3, 12, 291, 5)$
$(487, 239, 96, 223, 2)$
$(499, 340, 468, 272, 4)$
$(499, 193, 425, 380, 2)$
$(499, 321, 330, 218, 3)$
$(523, 128, 346, 67, 4)$
$(523, 479, 3, 45, 1)$
$(523, 427, 38, 333, 2)$
$(547, 407, 352, 418, 2)$
$(547, 407, 244, 180, 4)$
$(547, 339, 323, 432, 3)$
$(571, 474, 417, 251, 2)$
$(571, 91, 343, 298, 1)$
$(571, 91, 343, 298, 5)$
$(607, 3, 238, 39, 1)$
$(607, 3, 125, 430, 4)$
$(607, 317, 435, 74, 2)$
$(619, 2, 321, 75, 5)$

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