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Pursuing Analogies Between Differential Equations and Difference Equations

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1. Introduction. The study of ordinary differential equations has long been a staple in mathematics at both the undergraduate and graduate levels. Recently, instruction in the study of difference equations has widened, primarily due to the expanded role of the digital computer in mathematics. The two topics are inextricably linked at all levels, from elementary techniques through current research questions. Pursuing the analogies between these fields of study can only deepen the understanding of each. In particular, the study of many elementary topics in difference equations, requiring not even the use of calculus, can serve as a foundation for intuition and understanding of the analogous topics in differential equations. Since typical difficulties encountered at the introductory level in studying differential equations include development of intuition and avoiding the approach of pure memorization of formulas, such a foundation is indeed useful.

The purpose of this article is to illustrate how the analogy might be pursued through some typical problems and to comment on the usefulness of this analogy in the study of other topics. The comments here are, of course, not intended to be exhaustive; the goal is to suggest another way of thinking along with the traditional techniques for studying differential equations.

Note here that a thorough introduction to linear difference equations is provided by Miller [**5**], while Cadzow [**1**], Charlton [**2**], May [**4**], and others have described some of the dynamics and applications of first-order and higher-order difference equations.

2. Initial-value problems. Consider

$$y_{n+1} = F(n, y_n), \qquad n = 0, 1, 2, \ldots, \tag{1}$$

and

$$\dot{y}(t) = G(t, y(t)) \tag{2}$$

for t in some interval containing 0. To the student encountering the ordinary differential equation (2) for the first time, the notion that it may have many solutions, and that we appeal to a so-called initial value $y(0) = y_0$ to distinguish among solutions, is often foreign. On the other hand, the difference equation (1) can very easily be shown, by example, to exhibit such properties. A simple equation such as $y_{n+1} = y_n^2$ or $y_{n+1} = \cos y_n$ can quickly be investigated using a calculator. It will be apparent that many solution sequences exist (and can have widely varying behaviors); the initial value y_0 in (1) is needed to specify a particular solution. Thus the existence and uniqueness of the solution to the initial-value problem (IVP) consisting of (1) along with a given value of y_0 is completely obvious. A student will, immediately and intuitively, understand "starting at y_0" and moving from there as determined by the equation. By appeal to this line of thinking, the basic idea of a standard existence-uniqueness result for the IVP consisting of (2) with a given value $y(0) = y_0$ is plausible even to the complete novice; one "starts at y_0" and moves from there as determined by (2).

In an advanced course, the question of the maximal interval of existence of the solution to an IVP for (2) will be addressed. To establish intuition, we may again appeal to a simple example in (1); for instance, the IVP

$$y_{n+1} = \frac{2}{(1 - y_n)}, \qquad y_0 = 3,$$

generates terms $y_1 = -1$, $y_2 = 1$, and its solution then fails to exist further because the value $y_2 = 1$ is not in the domain of the right-hand side of the equation. Since the standard results concerning the interval of existence of the solution to an IVP for (2) essentially center on whether the solution approaches the boundary of the domain of the right-hand side of (2), the analogy is helpful.

At any level of study, autonomous versions of (1) and (2), namely,

$$y_{n+1} = f(y_n) \tag{3}$$

and

$$\dot{y}(t) = g(y(t)) \tag{4}$$

are of special importance. Note here that one may again appeal to simple examples in (1) and (3) to illustrate the contrast between autonomous and nonautonomous equations and hence to carry that idea to (2) and (4).

3. Linear equations. In an introductory study of differential equations, the solution of first-order linear equations is a standard early topic. Because students are unfamiliar with the methods involved and sometimes have not retained all of the subtleties from calculus, they often merely memorize some of the formulas that are presented. The corresponding study in difference equations is simpler and more easily remembered and therefore provides a useful foundation.

We might consider

$$y_{n+1} = ay_n + p_n \tag{5}$$

and

$$\dot{y}(t) = by(t) + q(t), \tag{6}$$

where a and b are constants, $\{p_n\}$ is a given sequence, and q is a given function. The solution to an IVP for (5) can be obtained by direct calculation and guessing: from $y_1 = ay_0 + p_0$, $y_2 = a^2y_0 + ap_0 + p_1$, etc., we predict that

$$y_n = a^n y_0 + \sum_{k=0}^{n-1} a^{n-1-k} p_k \tag{7}$$

and then verify that this is the correct formula. Whatever method and terminology is used to solve the IVP consisting of (6) along with a given value $y(0) = y_0$, the formula

$$y(t) = e^{bt}y_0 + \int_0^t e^{b(t-s)}q(s)\,ds \tag{8}$$

is obtained. The similarity between (7) and (8) is striking and useful. For example, the numbers a and e^b share a common role and the summation and convolution-type integral bear a powerful resemblance to one another. The details of the correspondence may be pursued at length; one may address homogeneous equations, stability, the role of the initial value, and so on. The importance of pointing out the correspondences lies in the fact that (5) and (7) are so simple to understand.

4. Linear vs. nonlinear. At an introductory level, even very able students can be slow to appreciate the depth of the contrast between problems involving linear differential equations and those involving nonlinear ones. A few examples of nonlinear difference equations can illustrate the fundamental points that need to be appreciated. To list only a few here, we note that closed-form solutions may not exist or may be elusive, growth or decay may not be of exponential nature, equilibria may have various properties of attraction or repulsion, and so on. Examples such as $y_{n+1} = k_1 y_n(k_2 - y_n)$, k_1, k_2 constant, suggest these ideas and more.

5. Higher-order equations and advanced topics. As with (1) and (2), we may compare

$$y_{n+2} = H(n, y_n, y_{n+1}) \tag{9}$$

with

$$\ddot{y}(t) = K(t, y(t), \dot{y}(t)) \tag{10}$$

and we can study pairs of higher-order equations. We motivate the appropriate IVP for (10) by appealing to specific examples of (9). The difficulty of another problem involving (10)—say, a two-point boundary value problem—can also be illustrated by using (9).

The standard approach to studying (10) begins with linear equations and includes work on two-dimensional first-order systems. Both have appropriate analogues among difference equations. We might compare

$$ay_{n+2} + by_{n+1} + cy_n = p_n \tag{11}$$

with

$$\alpha\ddot{y}(t) + \beta\dot{y}(t) + \gamma y(t) = q(t) \tag{12}$$

where a, b, c, α, β, and γ are constants, $\{p_n\}$ is a given sequence, and q is a given function. The pursuit of solutions to the homogeneous versions of (11) and (12)—of the form $\{r^n\}$ for (11) and e^{st} for (12), based on first-order experience—leads to characteristic equations for each. Conclusions about behavior of solutions to the homogeneous equations can be made based on the nature of the roots of the characteristic polynomials. The importance of linearity and the appropriateness of the search for one particular solution of the nonhomogeneous equation is identical in each study. We must note, however, that the detailed study of (11) is not much easier than that of (12); for instance, the concept of linear independence for sequences is really no simpler than for functions of a real variable.

If, however, (11) and (12) (or their higher-order cousins) are rewritten as systems of first-order equations, namely,

$$Y_{n+1} = AY_n + P_n \tag{13}$$

and

$$\dot{Y}(t) = BY(t) + Q(t), \tag{14}$$

where $\{Y_n\}$ and $\{P_n\}$ are sequences of vectors, Y and Q are vector-valued functions, and A and B are square matrices of constants, the advantages of keeping difference equations in mind re-emerge. The IVP solution formulas

$$Y_n = A^n Y_0 + \sum_{k=0}^{n-1} A^{n-1-k} P_k \tag{15}$$

and

$$Y(t) = e^{Bt} Y_0 + \int_0^t e^{B(t-s)} Q(s)\, ds \tag{16}$$

are obtained exactly as were (7) and (8). The comparison of A with e^B can be very useful to the student encountering matrix exponentials for the first time.

The classical topic of planar systems, beginning with homogeneous two-dimensional versions of (13) and (14), opens the door to the study of some advanced topics (cf. [**3**]). As graphical analyses are made (computer displays are, of course, extremely useful here) and emphasis is directed toward the role of the eigenvalues of A and B, a simple but often overlooked fact emerges quite clearly to the student: each solution curve of $\dot{Y}(t) = BY(t)$ passes through the points of corresponding solution sequences of $Y_{n+1} = e^B Y_n$. Thus, we may introduce the notion of the so-called Poincaré map, used to study properties of solutions of differential equations by studying certain sequences of points on those solutions.

I illustrate the preceding comment with an example. Take

$$B = \begin{bmatrix} -.8 & -.4 \\ 0 & -.4 \end{bmatrix}, \qquad e^B = \begin{bmatrix} e^{-.8} & e^{-.8} - e^{-.4} \\ 0 & e^{-.4} \end{bmatrix}.$$

Each solution sequence of $Y_{n+1} = e^B Y_n$ lies along a solution curve of $\dot{Y}(t) = BY(t)$ and hence provides a way for students to investigate and compare the dynamics of the systems. In Fig. 1 we have taken $Y_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and plotted Y_1 through Y_8 as well; the "shape" and "speed" of the approach to the origin emerge. The computations are simple even if a calculator is the only available tool. (We note an interesting irony related to graphical displays for solutions of differential equations: since a computer necessarily uses an approximating difference equation, why should we hesitate to use a difference equation whose solution sequences coincide *exactly* with sequences of points on the desired curves?)

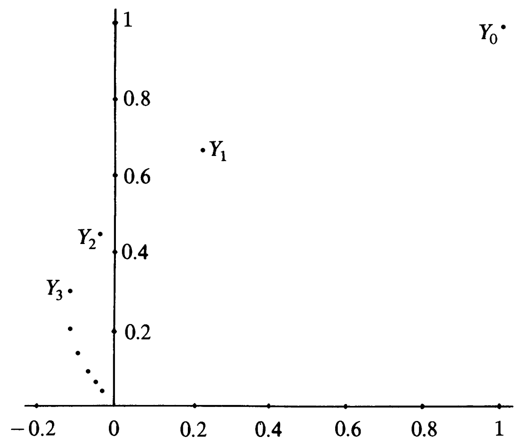

FIG. 1

The array of topics to be addressed in a more advanced course is wide and is a matter of the instructor's preference. Although more experienced students are in less need of support for their intuition, the parallel study of difference equations along with differential equations remains of pedagogical use and is of interest in its own right. Many topics in differential equations have direct analogues in difference equations, and, as has been suggested here, the corresponding study may well be simple and rewarding.

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On the Use of Iteration Methods for Approximating the Natural Logarithm

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1. Introduction. Usually one generates approximations to functions such as $\log(x)$ and $\exp(x)$ by first doing some kind of range reduction and then employing a polynomial or rational approximation. Reference [2] gives a number of examples of this, some of which have been actually used on some computers [3]. In this note